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A tripling construction for mutually orthogonal symmetric hamiltonian double Latin squares

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Abstract

We provide two new constructions for pairs of mutually orthogonal symmetric hamiltonian double Latin squares. The first is a tripling construction, and the second is derived from known constructions of hamilton cycle decompositions of K_p when p is prime.

KEYWORDS

complete graph, graph decomposition, hamilton cycle, Latin square

1 | INTRODUCTION

Latin squares with certain hamiltonian properties have many applications in graph theory. Wanless [11] used pan-hamiltonian Latin squares to build perfect 1-factorizations of complete graphs and complete bipartite graphs, where a Latin square is *pan-hamiltonian* if the permutation formed by any two rows is a single cycle. He later continued this study with Bryant and Maenhaut [3], where the name *pan-hamiltonian* was changed to *row-hamiltonian*. Grannell and Griggs [7] used *consecutively row-hamiltonian* Latin squares, where only adjacent rows are required to form a cyclic permutation, to build many different triangular embeddings of complete graphs and complete tripartite graphs in nonorientable surfaces; this was later extended to the orientable case by Grannell and Knor [8]. Choi and Chung [4] used a Latin square of order n constructed from a hamilton cycle in K_n to construct a routing algorithm for messages sent on a generalized recursive circulant network; this Latin square is consecutively row-hamiltonian for all n and row-hamiltonian for n prime. The current author and Ellingham used a pair of orthogonal Latin squares, one of which was required to satisfy a hamiltonian property on its entries, to build embeddings of complete tripartite graphs in orientable surfaces where each face is bounded by a hamilton cycle [6].

This connection between Latin squares and graph theory was extended to double Latin squares by Hilton et al [9]. There it was shown that a symmetric hamiltonian double Latin square of order $2n$ corresponds to a decomposition of the complete graph K_{2n} into hamilton paths, and that the decompositions arising from mutually orthogonal symmetric hamiltonian double Latin squares of order $2n$, hereafter referred to as MOSHLS($2n$), are themselves

orthogonal. In Hilton [9], a pair of MOSHLS($2n$) was constructed for all $n = 2^{\alpha}m$, where $\alpha \geq 0$ and $1 \leq m \leq 13$ is odd, but it remains to be determined whether or not a pair of MOSHLS($2n$) exists for all positive integers n. In this paper, we provide two new constructions for MOSHLS ($2n$): the first is a tripling construction, while the second is a direct construction derived from known hamilton cycle decompositions of K_p for p prime. Based on results obtained with these constructions, we formally set forth the following conjecture implied in Hilton [9].

Conjecture 1 *There exists a pair of MOSHLS($2n$) for all $n \geq 1$.*

Section 1.1 contains the relevant definitions for double Latin squares; further information on the connection between designs and graph decompositions or embeddings can be found in Colbourn and Dinitz [5].

1.1 | Definitions

A *double Latin square of order* $2n$ is a $2n \times 2n$ array of n symbols such that each symbol appears twice in every row and twice in every column. We shall assume, unless otherwise stated, that the rows and columns of a double Latin square of order $2n$ are indexed by $\mathbb{Z}_{2n}$ and that the symbols are taken from $\mathbb{Z}_n$. Let L be a double Latin square of order $2n$; we let (r, s) denote the *cell* located at the intersection of row r and column s, and we write $L_{r,s} = k$ if k is the symbol that appears in cell (r, s). For any symbol $k \in \mathbb{Z}_n$, define the graph $G_L(k)$ with vertex set $\{\alpha_0, \alpha_1, ..., \alpha_{2n-1}, \beta_0, \beta_1, ..., \beta_{2n-1}\}$ and edge set $\{\alpha_r\beta_s \mid L_{r,s} = k\}$; we will refer to $\alpha_r\beta_s$ as the edge corresponding to the cell (r, s), and vice versa. Clearly $G_L(k)$ is a bipartite 2-regular graph, and we say L is *hamiltonian* if for every symbol $k \in \mathbb{Z}_n$ the graph $G_L(k)$ is a single (hamilton) cycle of length $4n$; we will let HLS($2n$) denote a hamiltonian double Latin square of order $2n$. Two double Latin squares L and M on the same set of symbols are *orthogonal* if for every ordered pair of symbols (k, ℓ) there exist precisely four ordered pairs (r, s) such that $L_{r,s} = k$ and $M_{r,s} = \ell$. Finally, a double Latin square L is *symmetric* if $L_{r,s} = L_{s,r}$ for all $r, s \in \mathbb{Z}_{2n}$. It was shown in Hilton [9] that every symbol must appear twice on the main diagonal of a symmetric HLS($2n$); if the main diagonal of a symmetric HLS($2n$) L is of the form $(0, 1, ..., n-1, 0, 1, ..., n-1)$ we say L is in *normal form*. According to Hilton [9, Proposition 5.3], we may assume for the remainder of the paper that, unless otherwise stated, a symmetric HLS($2n$) is in normal form, and we shall refer to the cells along the main diagonal as *diagonal cells*.

A *decomposition* of a graph G is a collection $\mathcal{H}$ of disjoint subgraphs of G such that every edge of G appears in precisely one subgraph $H \in \mathcal{H}$; if each $H \in \mathcal{H}$ is a hamilton path (resp., hamilton cycle) in G, then $\mathcal{H}$ is called a *hamilton path decomposition* (resp., *hamilton cycle decomposition*) of G. Let $G = K_{2n}$ for some $n \geq 1$; from Hilton [9, Lemma 5.5] we say two hamilton path decompositions $\mathcal{H} = \{H_0, H_1, ..., H_{n-1}\}$ and $\mathcal{H}' = \{H'_0, H'_1, ..., H'_{n-1}\}$ of K_{2n} are *orthogonal* if

1. H_k and H'_k have the same endvertices for each $k \in \mathbb{Z}_n$,
2. $|E(H_k) \cap E(H'_k)| = 1$ for all $k \in \mathbb{Z}_n$,
3. $|E(H_k) \cap E(H'_\ell)| = 2$ for all $k \neq \ell \in \mathbb{Z}_n$.

Let $V(K_{2n}) = \{u_0, u_1, ..., u_{2n-1}\}$ be the vertex set of the complete graph K_{2n}, and let L be a symmetric HLS($2n$). For each $k \in \mathbb{Z}_n$, let H_k be the subgraph of K_{2n} with edge set $E(H_k) = \{u_r u_s | L_{r,s} = k, r \neq s\}$. It was shown in Hilton [9] that the collection

$\mathcal{H} = \{H_0, H_1, ..., H_{n-1}\}$ is a hamilton path decomposition of K_{2n}. Moreover, two symmetric HLS $(2n)$ L and L' are orthogonal if and only if their corresponding hamilton path decompositions are orthogonal [9, Lemma 5.4].

2 | A TRIPLING CONSTRUCTION

Suppose L and M are a pair of MOSHLS$(2n)$. For every $k \in \mathbb{Z}_n$, let $r_k \neq s_k \in \mathbb{Z}_{2n}$ be such that $L_{r_k,s_k} = M_{r_k,s_k} = k$, so that the four cells containing the symbol k in both L and M are (k, k), $(n + k, n + k)$, (r_k, s_k), (s_k, r_k). We 2-color the cells in L containing k with red and blue such that every row and every column contains one blue cell and one red cell; this is equivalent to a proper 2-edge-coloring of $G_L(k)$ and is unique up to isomorphism. We call such a coloring a *k-cell 2-coloring of L*. By the symmetry of L, the cells (k, k) and $(n + k, n + k)$ must receive the same color, likewise for (r_k, s_k) and (s_k, r_k). If for all $k \in \mathbb{Z}_n$ the cells (k, k) and (r_k, s_k) receive different colors in a k-cell 2-coloring of L, then we say L is *diagonally distinguished with respect to M*.

Theorem 2.1 *If there exists a pair of MOSHLS$(2n)$ L and M such that L is diagonally distinguished with respect to M, then there exists a pair of MOSHLS$(6n)$ $\widehat{L}$ and $\widehat{M}$ such that $\widehat{L}$ is diagonally distinguished with respect to $\widehat{M}$.*

Proof Assume that L and M are a pair of MOSHLS$(2n)$ such that L is diagonally distinguished with respect to M; for all $k \in \mathbb{Z}_n$, take a k-cell 2-coloring of L that assigns diagonal cells the color red. For convenience, we will usually denote the element $(a, b) \in A \times B$ as a^b. Given an array A, we let $k \times A$ denote the array where each symbol a appearing in A is replaced by k^a. Let A, B, C, A', and B' be the following 3×3 arrays:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix},$$
$$A' = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \quad B' = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Note that A, B, and C are Latin squares.

For every cell colored red, we replace the symbol k in L with $k \times A'$; for every cell colored blue, we replace the symbol k with $k \times B'$. Complete this process for all $k \in \mathbb{Z}_n$, and call the resulting $6n \times 6n$ array $\widehat{L}$. Since every row and every column contain one copy of $k \times A'$ and one copy of $k \times B'$, it follows that $\widehat{L}$ is a double Latin square; moreover, A' and B' are symmetric, therefore $\widehat{L}$ is also symmetric.
For every $k \in \mathbb{Z}_n$, denote the graph $G_M(k)$ by

$$(v_0\ w_0\ v_1\ w_1 \cdots v_{2n-1}\ w_{2n-1}),$$

such that, according to the notation given in Section 1.1, $v_0 w_0 = \alpha_k \beta_k$ and $v_n w_n = \alpha_{n+k} \beta_{n+k}$. Without loss of generality, take (r_k, s_k) to be the nondiagonal cell containing k in both L and M and satisfying $v_{i_k} = \alpha_{r_k}$ for some $i_k \in \mathbb{Z}_n \backslash \{0\}$ (in other

words, the cell (r_k, s_k) corresponds to an edge in the first half of $G_M(k)$ as written above). For all cells corresponding to edges in the path

$$[v_0\ w_0\ v_1\ w_1 \cdots w_{n-1}\ v_n]$$

of $G_M(k)$, replace the symbol k in M with $k \times C$, except

- replace k in cell (k, k) with $k \times A$,
- replace k in cell (r_k, s_k) with $k \times B$.

For all cells corresponding to edges in the path

$$[v_n\ w_n\ v_{n+1}\ w_{n+1} \cdots w_{2n-1}\ v_0]$$

of $G_M(k)$, replace the symbol k in M with $k \times C^T$, except

- replace k in cell $(n + k, n + k)$ with $k \times A$,
- replace k in cell (s_k, r_k) with $k \times B$.

Complete this process for all $k \in \mathbb{Z}_n$, and call the resulting $6n \times 6n$ array $\widehat{M}$. Since A, B, and C are Latin squares, $\widehat{M}$ is clearly a double Latin square. Moreover, A and B are symmetric, and for every occurrence of $k \times C$ in a cell (r, s) the symmetric cell (s, r) contains $k \times C^T$, so $\widehat{M}$ is symmetric. It remains to show that $\widehat{L}$ and $\widehat{M}$ form an orthogonal pair, that each square is hamiltonian, and that $\widehat{L}$ is diagonally distinguished with respect to $\widehat{M}$.

Consider first the ordered pair of symbols (k^a, k^b), where $k \in \mathbb{Z}_n$ and $a, b \in \mathbb{Z}_3$. We know that both L and M contain the symbol k in cells (k, k), $(n + k, n + k)$, (r_k, s_k), (s_k, r_k). Since L is diagonally distinguished with respect to M, we know the cells (k, k) and $(n + k, n + k)$ are colored red while (r_k, s_k) and (s_k, r_k) are colored blue. The red cells in L have been replaced by $k \times A'$, while the corresponding cells in M have been replaced by $k \times A$. As shown below, this pair of subsquares, which appears twice, covers the pairs (k^0, k^2), (k^1, k^0), (k^2, k^1) four times each and the pairs (k^0, k^0), (k^1, k^1), (k^2, k^2) twice each.

$$k \times A' = \begin{bmatrix} k^0 & k^0 & k^2 \\ k^0 & k^1 & k^1 \\ k^2 & k^1 & k^2 \end{bmatrix} k \times A = \begin{bmatrix} k^0 & k^2 & k^1 \\ k^2 & k^1 & k^0 \\ k^1 & k^0 & k^2 \end{bmatrix}.$$

The blue cells in L have been replaced by $k \times B'$, while the corresponding cells in M have been replaced by $k \times B$. As shown below, this pair of subsquares, which appears twice, covers the pairs (k^0, k^1), (k^1, k^2), (k^2, k^0) four times each and the pairs (k^0, k^0), (k^1, k^1), (k^2, k^2) twice each.

$$k \times B' = \begin{bmatrix} k^1 & k^2 & k^1 \\ k^2 & k^2 & k^0 \\ k^1 & k^0 & k^0 \end{bmatrix} k \times B = \begin{bmatrix} k^1 & k^0 & k^2 \\ k^0 & k^2 & k^1 \\ k^2 & k^1 & k^0 \end{bmatrix}.$$

Together every ordered pair (k^a, k^b), with $k \in \mathbb{Z}_n$ and $a, b \in \mathbb{Z}_3$, is covered precisely four times.

Consider next an ordered pair of symbols (k^a, ℓ^b) with $k \neq \ell \in \mathbb{Z}_n$ and $a, b \in \mathbb{Z}_3$. Since $k \neq \ell$, the four cells that contain k in L and ℓ in M must be replaced by $k \times A'$ or $k \times B'$ in L and $\ell \times C$ or $\ell \times C^T$ in M. By comparing $k \times A'$ and $k \times B'$ written above with $\ell \times C$ and $\ell \times C^T$ below, it is clear that any of the four possible pairings covers each ordered pair (k^a, ℓ^b) once.

$$\ell \times C = \begin{bmatrix} \ell^0 & \ell^1 & \ell^2 \\ \ell^2 & \ell^0 & \ell^1 \\ \ell^1 & \ell^2 & \ell^0 \end{bmatrix} \ell \times C^T = \begin{bmatrix} \ell^0 & \ell^2 & \ell^1 \\ \ell^1 & \ell^0 & \ell^2 \\ \ell^2 & \ell^1 & \ell^0 \end{bmatrix}.$$

Together the four cells that cover (k, ℓ) in L and M are replaced with squares that cover each pair (k^a, ℓ^b), with $a, b \in \mathbb{Z}_3$, precisely four times as required. Thus, $\widehat{L}$ and $\widehat{M}$ are orthogonal.

Let $k^a \in \mathbb{Z}_n \times \mathbb{Z}_3$; we must show that both $G_{\widehat{L}}(k^a)$ and $G_{\widehat{M}}(k^a)$ consist of a single cycle of length $12n$. Let

$$G_L(k) = (x_0\ y_0\ x_1\ y_1 \cdots x_{2n-1}\ y_{2n-1}),$$

and assume that $x_0 y_0$ (and thus $x_i y_i$ for every $i \in \mathbb{Z}_{2n}$) corresponds to a red cell in the k-cell 2-coloring of L used previously. Every row r and column c in L yields three rows and columns, respectively, in $\widehat{L}$; we label these rows and columns r^0, r^1, r^2 and c^0, c^1, c^2, respectively, so that the cell (r^b, s^c) contains the symbol k^a if and only if $k = L_{r,s}$ and $a = D_{b,c}$, where $D = A'$ or B' as appropriate. Furthermore, if the edge $x_i y_j \in G_L(k)$ corresponds to a cell (r, s) containing k, then the edge corresponding to the cell (r^b, s^c) containing k^a will be written $x_i^b y_j^c \in G_{\widehat{L}}(k^a)$. Using this notation, we see that for any edge $x_i y_i \in G_L(k)$ corresponding to a red cell in L, we obtain the path $P_i^a = [x_i^{a+1}\ y_i^a\ x_i^a\ y_i^{a+1}]$ in $G_{\widehat{L}}(k^a)$, and for any edge $y_i x_{i+1}$ corresponding to a blue cell in L, we obtain the path $Q_i^a = [y_i^{a+1}\ x_{i+1}^{a+2}\ y_i^{a+2}\ x_{i+1}^{a+1}]$ in $G_{\widehat{L}}(k^a)$. Identifying common endvertices of these paths, we see that

$$\begin{aligned} G_{\widehat{L}}(k^a) &= \big(x_0^{a+1}\ y_0^a\ x_0^a\ y_0^{a+1}\ x_1^{a+2}\ y_0^{a+2}\ x_1^{a+1}\ y_1^a\ x_1^a\ y_1^{a+1}\ x_2^{a+2}\ y_1^{a+2}\ x_2^{a+1} \cdots \\ &\quad x_{2n-1}^{a+1}\ y_{2n-1}^a\ x_{2n-1}^a\ y_{2n-1}^{a+1}\ x_0^{a+2}\ y_{2n-1}^{a+2}\big), \\ &= \big(P_0^a\ Q_0^a\ P_1^a\ Q_1^a \cdots P_{2n-1}^a\ Q_{2n-1}^a\big), \end{aligned}$$

which is a cycle of length $12n$. Hence, $\widehat{L}$ is hamiltonian.

Consider again

$$G_M(k) = (v_0\ w_0\ v_1\ w_1 \cdots v_{2n-1}\ w_{2n-1}),$$

and recall that $v_{i_k} = \alpha_{r_k}$ for some $i_k \in \mathbb{Z}_n \backslash \{0\}$, where $L_{r_k,s_k} = M_{r_k,s_k} = k$; this immediately implies by the symmetry of $G_M(k)$ that $w_{2n-i_k} = \beta_{r_k}$. We must also have either $w_{i_k} = \beta_{s_k}$ and $v_{2n-i_k} = \alpha_{s_k}$ or $w_{i_k-1} = \beta_{s_k}$ and $v_{2n-i_k+1} = \alpha_{s_k}$; let $i_k^* = i_k$ or $i_k^* = i_k - 1$ accordingly, so that $w_{i_k^*} = \beta_{s_k}$. Let $a \in \mathbb{Z}_3$; using notation similar to the previous paragraph, we determine $G_{\widehat{M}}(k^a)$ for the following two cases, noting that all arithmetic is performed modulo 3 for superscripts and modulo $2n$ for subscripts:

Case 1. $i_k^* = i_k$. We first partition the edges of $G_M(k)$ into the four subpaths R_1, R_2, R_3, and R_4 shown below:

$$\begin{aligned} R_1 &= [w_0\ v_1\ w_1 \cdots v_{i_k}\ w_{i_k}], \\ R_2 &= [w_{i_k}\ v_{i_k+1}\ w_{i_k+1} \cdots v_n\ w_n], \\ R_3 &= [w_n\ v_{n+1}\ w_{n+1} \cdots v_{2n-i_k}\ w_{2n-i_k}], \\ R_4 &= [w_{2n-i_k}\ v_{2n-i_k+1}\ w_{2n-i_k+1} \cdots v_0\ w_0]. \end{aligned}$$

We want to find the "lifts" of each of these paths in $G_{\widehat{M}}(k^a)$. Every edge vw appearing in R_1 and R_2, except the final edge in each, corresponds to a cell replaced with $k \times C$ and is thus lifted to the edges $v^j w^{j+a}$ for $j \in \mathbb{Z}_3$. Similarly, every edge vw appearing in R_3 and R_4, except the final edge in each, corresponds to a cell replaced with $k \times C^T$ and is thus lifted to the edges $v^j w^{j-a}$ for $j \in \mathbb{Z}_3$. The final edge vw in each of R_1 and R_3 corresponds to a cell replaced with $k \times B$ and is thus lifted to the edges $v^j w^{2a-j+1}$ for $j \in \mathbb{Z}_3$, whereas the final edge vw in each of R_2 and R_4 corresponds to a cell replaced with $k \times A$ and is thus lifted to the edges $v^j w^{2a-j}$ for $j \in \mathbb{Z}_3$. Each subpath R_h above is lifted to three subpaths R_h^j of $G_{\widehat{M}}(k^a)$, where $h = 1, 2, 3, 4$ and $j \in \mathbb{Z}_3$. These subpaths are given below:

$$\begin{aligned} R_1^j &= [w_0^{j+a}\ v_1^j\ w_1^{j+a} \cdots v_{i_k}^j\ w_{i_k}^{2a-j+1}], \\ R_2^j &= [w_{i_k}^{j+a}\ v_{i_k+1}^j\ w_{i_k+1}^{j+a} \cdots v_n^j\ w_n^{2a-j}], \\ R_3^j &= [w_n^{j-a}\ v_{n+1}^j\ w_{n+1}^{j-a} \cdots v_{2n-i_k}^j\ w_{2n-i_k}^{2a-j+1}], \\ R_4^j &= [w_{2n-i_k}^{j-a}\ v_{2n-i_k+1}^j\ w_{2n-i_k+1}^{j-a} \cdots v_0^j\ w_0^{2a-j}]. \end{aligned}$$

Finally, we must join these paths by identifying common endpoints to form $G_{\widehat{M}}(k^a)$:

$$G_{\widehat{M}}(k^a) = \left(R_1^0\ R_2^{a+1}\ R_3^{2a+2}\ R_4^{a+2}\ R_1^1\ R_2^a\ R_3^{2a}\ R_4^{a+1}\ R_1^2\ R_2^{a+2}\ R_3^{2a+1}\ R_4^a\right).$$

Case 2. $i_k^* = i_k - 1$. We partition the edges of $G_M(k)$ into the four subpaths S_1, S_2, S_3, and S_4 shown below:

$$\begin{aligned} S_1 &= [w_0\ v_1\ w_1 \cdots w_{i_k-1}\ v_{i_k}], \\ S_2 &= [v_{i_k}\ w_{i_k}\ v_{i_k+1} \cdots v_n\ w_n], \\ S_3 &= [w_n\ v_{n+1}\ w_{n+1} \cdots w_{2n-i_k}\ v_{2n-i_k+1}], \\ S_4 &= [v_{2n-i_k+1}\ w_{2n-i_k+1}\ v_{2n-i_k+2} \cdots v_0\ w_0]. \end{aligned}$$

As before we determine the lifts of each of these paths in $G_{\widehat{M}}(k^a)$. The only changes from Case 1 are that the edges $v_{i_k} w_{i_k}$ and $v_{2n-i_k} w_{2n-i_k}$ correspond to cells replaced with $k \times C$ and $k \times C^T$, respectively, while the edges $w_{i_k-1} v_{i_k}$ and $w_{2n-i_k} v_{2n-i_k+1}$ correspond to cells replaced with $k \times B$. The resulting lifts are shown below:

$$\begin{aligned}
S_1^j &= [w_0^{j+a}\ v_1^j\ w_1^{j+a} \cdots w_{i_k-1}^{j+a}\ v_{i_k}^{a-j+1}],\\
S_2^j &= [v_{i_k}^j\ w_{i_k}^{j+a}\ v_{i_k+1}^j \cdots v_n^j\ w_n^{2a-j}],\\
S_3^j &= [w_n^{j-a}\ v_{n+1}^j\ w_{n+1}^{j-a} \cdots w_{2n-i_k}^{j-a}\ v_{2n-i_k+1}^{2j+1}],\\
S_4^j &= [v_{2n-i_k+1}^j\ w_{2n-i_k+1}^{j-a}\ v_{2n-i_k+2}^j \cdots v_0^j\ w_0^{2a-j}].
\end{aligned}$$

Identifying common endpoints yields:

$$G_{\widehat{M}}(k^a) = (S_1^0\ S_2^{a+1}\ S_3^{2a+2}\ S_4^{a+2}\ S_1^1\ S_2^a\ S_3^{2a}\ S_4^{a+1}\ S_1^2\ S_2^{a+2}\ S_3^{2a+1}\ S_4^a).$$

Since each case produces a single cycle of length $12n$, we know $\widehat{M}$ is hamiltonian. Finally, define the following k^a-cell 2-coloring of $\widehat{L}$: take any subsquare $k \times A'$, which replaced a red cell from L, and color the diagonal cell containing k^a red and the two nondiagonal cells containing k^a blue. Moreover, the blue cells appearing in the same row or column as the original red cell from L have been replaced by $k \times B'$; color the two nondiagonal cells containing k^a red and the diagonal cell containing k^a blue. This coloring extends to every copy of $k \times A'$ and $k \times B'$ appearing in $\widehat{L}$. Recall that the cells (k, k), $(n + k, n + k)$, (r_k, s_k), and (s_k, r_k) contain the symbol k in both L and M. Since A' and A both have $(0, 1, 2)$ along their main diagonal and B' and B both have $(1, 2, 0)$ along their main diagonal, we know (k^a, k^a), $((n + k)^a, (n + k)^a)$, (r_k^{a-1}, s_k^{a-1}), and (s_k^{a-1}, r_k^{a-1}) are the cells containing k^a in both $\widehat{L}$ and $\widehat{M}$. Since (k^a, k^a) and $((n + k)^a, (n + k)^a)$ are diagonal cells of a $k \times A'$ subsquare, they are colored red; since (r_k^{a-1}, s_k^{a-1}) and (s_k^{a-1}, r_k^{a-1}) are diagonal cells of a $k \times B'$ subsquare, they are colored blue. Thus, $\widehat{L}$ is diagonally distinguished with respect to $\widehat{M}$. □

2.1 | Example

We apply the tripling construction of Theorem 2.1 to the following pair of MOSHLS(6) L and M (A_3 and B_3, respectively, from Hilton [9]):

$$L = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 2 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 2 & 1 \\ 2 & 0 & 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 2 & 2 \\ 0 & 2 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 & 2 \end{bmatrix},$$

$$M' = \begin{bmatrix} A & C & C & B & C^T & C^T \\ C^T & A & C^T & C^T & B & C^T \\ C^T & C & A & C^T & C^T & B \\ B & C & C & A & C^T & C^T \\ C & B & C & C & A & C \\ C & C & B & C & C^T & A \end{bmatrix}.$$

The square M' contains the subsquares used to construct $\widehat{M}$, so that $M'_{r,s} = D$ if and only if the entry k in cell (r, s) of M is replaced by $k \times D$, for $D \in \{A, B, C, C^T\}$. For $\widehat{L}$, entry k in cell (r, s) of L is replaced by $k \times A'$ if $r + s$ is even, and $k \times B'$ otherwise. The resulting MOSHLS(18) $\widehat{L}$

and $\widehat{M}$ are shown below:

$$\widehat{L} = \left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0^0 & 0^0 & 0^2 & 0^1 & 0^2 & 0^1 & 1^0 & 1^0 & 1^2 & 1^1 & 1^2 & 1^1 & 2^0 & 2^0 & 2^2 & 2^1 & 2^2 & 2^1 \\
0^0 & 0^1 & 0^1 & 0^2 & 0^2 & 0^0 & 1^0 & 1^1 & 1^1 & 1^2 & 1^2 & 1^0 & 2^0 & 2^1 & 2^1 & 2^2 & 2^2 & 2^0 \\
0^2 & 0^1 & 0^2 & 0^1 & 0^0 & 0^0 & 1^2 & 1^1 & 1^2 & 1^1 & 1^0 & 1^0 & 2^2 & 2^1 & 2^2 & 2^1 & 2^0 & 2^0 \\
\hline
0^1 & 0^2 & 0^1 & 1^0 & 1^0 & 1^2 & 1^1 & 1^2 & 1^1 & 2^0 & 2^0 & 2^2 & 2^1 & 2^2 & 2^1 & 0^0 & 0^0 & 0^2 \\
0^2 & 0^2 & 0^0 & 1^0 & 1^1 & 1^1 & 1^2 & 1^2 & 1^0 & 2^0 & 2^1 & 2^1 & 2^2 & 2^2 & 2^0 & 0^0 & 0^1 & 0^1 \\
0^1 & 0^0 & 0^0 & 1^2 & 1^1 & 1^2 & 1^1 & 1^0 & 1^0 & 2^2 & 2^1 & 2^2 & 2^1 & 2^0 & 2^0 & 0^2 & 0^1 & 0^2 \\
\hline
1^0 & 1^0 & 1^2 & 1^1 & 1^2 & 1^1 & 2^0 & 2^0 & 2^2 & 2^1 & 2^2 & 2^1 & 0^0 & 0^0 & 0^2 & 0^1 & 0^2 & 0^1 \\
1^0 & 1^1 & 1^1 & 1^2 & 1^2 & 1^0 & 2^0 & 2^1 & 2^1 & 2^2 & 2^2 & 2^0 & 0^0 & 0^1 & 0^1 & 0^2 & 0^2 & 0^0 \\
1^2 & 1^1 & 1^2 & 1^1 & 1^0 & 1^0 & 2^2 & 2^1 & 2^2 & 2^1 & 2^0 & 2^0 & 0^2 & 0^1 & 0^2 & 0^1 & 0^0 & 0^0 \\
\hline
1^1 & 1^2 & 1^1 & 2^0 & 2^0 & 2^2 & 2^1 & 2^2 & 2^1 & 0^0 & 0^0 & 0^2 & 0^1 & 0^2 & 0^1 & 1^0 & 1^0 & 1^2 \\
1^2 & 1^2 & 1^0 & 2^0 & 2^1 & 2^1 & 2^2 & 2^2 & 2^0 & 0^0 & 0^1 & 0^1 & 0^2 & 0^2 & 0^0 & 1^0 & 1^1 & 1^1 \\
1^1 & 1^0 & 1^0 & 2^2 & 2^1 & 2^2 & 2^1 & 2^0 & 2^0 & 0^2 & 0^1 & 0^2 & 0^1 & 0^0 & 0^0 & 1^2 & 1^1 & 1^2 \\
\hline
2^0 & 2^0 & 2^2 & 2^1 & 2^2 & 2^1 & 0^0 & 0^0 & 0^2 & 0^1 & 0^2 & 0^1 & 1^0 & 1^0 & 1^2 & 1^1 & 1^2 & 1^1 \\
2^0 & 2^1 & 2^1 & 2^2 & 2^2 & 2^0 & 0^0 & 0^1 & 0^1 & 0^2 & 0^2 & 0^0 & 1^0 & 1^1 & 1^1 & 1^2 & 1^2 & 1^0 \\
2^2 & 2^1 & 2^2 & 2^1 & 2^0 & 2^0 & 0^2 & 0^1 & 0^2 & 0^1 & 0^0 & 0^0 & 1^2 & 1^1 & 1^2 & 1^1 & 1^0 & 1^0 \\
\hline
2^1 & 2^2 & 2^1 & 0^0 & 0^0 & 0^2 & 0^1 & 0^2 & 0^1 & 1^0 & 1^0 & 1^2 & 1^1 & 1^2 & 1^1 & 2^0 & 2^0 & 2^2 \\
2^2 & 2^2 & 2^0 & 0^0 & 0^1 & 0^1 & 0^2 & 0^2 & 0^0 & 1^0 & 1^1 & 1^1 & 1^2 & 1^2 & 1^0 & 2^0 & 2^1 & 2^1 \\
2^1 & 2^0 & 2^0 & 0^2 & 0^1 & 0^2 & 0^1 & 0^0 & 0^0 & 1^2 & 1^1 & 1^2 & 1^1 & 1^0 & 1^0 & 2^2 & 2^1 & 2^2
\end{array}\right]$$

$$\widehat{M} = \left[\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc}
0^0 & 0^2 & 0^1 & 2^0 & 2^1 & 2^2 & 2^0 & 2^1 & 2^2 & 1^1 & 1^0 & 1^2 & 0^0 & 0^2 & 0^1 & 1^0 & 1^2 & 1^1 \\
0^2 & 0^1 & 0^0 & 2^2 & 2^0 & 2^1 & 2^2 & 2^0 & 2^1 & 1^0 & 1^2 & 1^1 & 0^1 & 0^0 & 0^2 & 1^1 & 1^0 & 1^2 \\
0^1 & 0^0 & 0^2 & 2^1 & 2^2 & 2^0 & 2^1 & 2^2 & 2^0 & 1^2 & 1^1 & 1^0 & 0^2 & 0^1 & 0^0 & 1^2 & 1^1 & 1^0 \\
\hline
2^0 & 2^2 & 2^1 & 1^0 & 1^2 & 1^1 & 0^0 & 0^2 & 0^1 & 0^0 & 0^2 & 0^1 & 2^1 & 2^0 & 2^2 & 1^0 & 1^2 & 1^1 \\
2^1 & 2^0 & 2^2 & 1^2 & 1^1 & 1^0 & 0^1 & 0^0 & 0^2 & 0^1 & 0^0 & 0^2 & 2^0 & 2^2 & 2^1 & 1^1 & 1^0 & 1^2 \\
2^2 & 2^1 & 2^0 & 1^1 & 1^0 & 1^2 & 0^2 & 0^1 & 0^0 & 0^2 & 0^1 & 0^0 & 2^2 & 2^1 & 2^0 & 1^2 & 1^1 & 1^0 \\
\hline
2^0 & 2^2 & 2^1 & 0^0 & 0^1 & 0^2 & 2^0 & 2^2 & 2^1 & 1^0 & 1^2 & 1^1 & 1^0 & 1^2 & 1^1 & 0^1 & 0^0 & 0^2 \\
2^1 & 2^0 & 2^2 & 0^2 & 0^0 & 0^1 & 2^2 & 2^1 & 2^0 & 1^1 & 1^0 & 1^2 & 1^1 & 1^0 & 1^2 & 0^0 & 0^2 & 0^1 \\
2^2 & 2^1 & 2^0 & 0^1 & 0^2 & 0^0 & 2^1 & 2^0 & 2^2 & 1^2 & 1^1 & 1^0 & 1^2 & 1^1 & 1^0 & 0^2 & 0^1 & 0^0 \\
\hline
1^1 & 1^0 & 1^2 & 0^0 & 0^1 & 0^2 & 1^0 & 1^1 & 1^2 & 0^0 & 0^2 & 0^1 & 2^0 & 2^2 & 2^1 & 2^0 & 2^2 & 2^1 \\
1^0 & 1^2 & 1^1 & 0^2 & 0^0 & 0^1 & 1^2 & 1^0 & 1^1 & 0^2 & 0^1 & 0^0 & 2^1 & 2^0 & 2^2 & 2^1 & 2^0 & 2^2 \\
1^2 & 1^1 & 1^0 & 0^1 & 0^2 & 0^0 & 1^1 & 1^2 & 1^0 & 0^1 & 0^0 & 0^2 & 2^2 & 2^1 & 2^0 & 2^2 & 2^1 & 2^0 \\
\hline
0^0 & 0^1 & 0^2 & 2^1 & 2^0 & 2^2 & 1^0 & 1^1 & 1^2 & 2^0 & 2^1 & 2^2 & 1^0 & 1^2 & 1^1 & 0^0 & 0^1 & 0^2 \\
0^2 & 0^0 & 0^1 & 2^0 & 2^2 & 2^1 & 1^2 & 1^0 & 1^1 & 2^2 & 2^0 & 2^1 & 1^2 & 1^1 & 1^0 & 0^2 & 0^0 & 0^1 \\
0^1 & 0^2 & 0^0 & 2^2 & 2^1 & 2^0 & 1^1 & 1^2 & 1^0 & 2^1 & 2^2 & 2^0 & 1^1 & 1^0 & 1^2 & 0^1 & 0^2 & 0^0 \\
\hline
1^0 & 1^1 & 1^2 & 1^0 & 1^1 & 1^2 & 0^1 & 0^0 & 0^2 & 2^0 & 2^1 & 2^2 & 0^0 & 0^2 & 0^1 & 2^0 & 2^2 & 2^1 \\
1^2 & 1^0 & 1^1 & 1^2 & 1^0 & 1^1 & 0^0 & 0^2 & 0^1 & 2^2 & 2^0 & 2^1 & 0^1 & 0^0 & 0^2 & 2^2 & 2^1 & 2^0 \\
1^1 & 1^2 & 1^0 & 1^1 & 1^2 & 1^0 & 0^2 & 0^1 & 0^0 & 2^1 & 2^2 & 2^0 & 0^2 & 0^1 & 0^0 & 2^1 & 2^0 & 2^2
\end{array}\right]$$

For the benefit of the reader, we trace out the graph $G_{\widehat{M}}(1^2)$. First, note that

$$G_M(1) = (\alpha_1\ \beta_1\ \alpha_5\ \beta_0\ \alpha_3\ \beta_2\ \alpha_4\ \beta_4\ \alpha_2\ \beta_3\ \alpha_0\ \beta_5).$$

Additionally note that $(r_1, s_1) = (3, 0)$, so that $i_1 = 2$ and $i_1^* = i_1 - 1 = 1$. Writing $\alpha_r^b\ \beta_s^c$ for the edge corresponding to cell (r^b, s^c), we obtain the graph

$$\begin{aligned} G_{\widehat{M}}(1^2) &= \big(\beta_1^2 \quad \alpha_5^0 \quad \beta_0^2 \quad \alpha_3^0 \quad \beta_2^2 \quad \alpha_4^0 \quad \beta_4^1 \quad \alpha_2^0 \quad \beta_3^1 \quad \alpha_0^1 \quad \beta_5^2 \quad \alpha_1^1 \\ &\quad\ \ \beta_1^0 \quad \alpha_5^1 \quad \beta_0^0 \quad \alpha_3^2 \quad \beta_2^1 \quad \alpha_4^2 \quad \beta_4^2 \quad \alpha_2^1 \quad \beta_3^2 \quad \alpha_0^0 \quad \beta_5^1 \quad \alpha_1^0 \\ &\quad\ \ \beta_1^1 \quad \alpha_5^2 \quad \beta_0^1 \quad \alpha_3^1 \quad \beta_2^0 \quad \alpha_4^1 \quad \beta_4^0 \quad \alpha_2^2 \quad \beta_3^0 \quad \alpha_0^2 \quad \beta_5^0 \quad \alpha_1^2\big), \\ &= \big(S_1^0 \quad S_2^0 \quad S_3^0 \quad S_4^1 \quad S_1^1 \quad S_2^2 \quad S_3^1 \quad S_4^0 \quad S_1^2 \quad S_2^1 \quad S_3^2 \quad S_4^2\big), \end{aligned}$$

as expected.

3 | A DIRECT CONSTRUCTION FROM HAMILTON CYCLE DECOMPOSITIONS OF COMPLETE GRAPHS

There are several well-known families of hamilton cycle decompositions of the complete graph K_n for odd n. The first construction was given by Lucas [10] and attributed to Walecki; it is discussed further in Alspach [2]. For our purposes, we will utilize two families of hamilton cycle decompositions of K_p, where p is prime, that appear in Akiyama et al [1]. We generally follow the author's notation in Akiyama et al [1], but replace $\mathbb{Z}_p$ with $\{u_i \mid i \in \mathbb{Z}_p\}$ as the vertex set of K_p. Let $p \geq 3$ be prime and set $r = (p-1)/2$; we define the decompositions $\mathcal{G}_p = \{G_1, G_2, ..., G_r\}$ and $\mathcal{B}_p = \{B_1, B_2, ..., B_r\}$ of K_p on vertex set $\{u_0, u_1, ..., u_{p-1}\}$ as follows. For all $1 \leq k \leq r$, let

$$G_k = (u_0\ u_k\ u_{2k} \cdots u_{(p-1)k}).$$

Given $1 \leq k \leq r$, define the following subsets of $E(K_p)$:

$$\begin{aligned} S_k &= \{u_a\ u_b \mid a + b \equiv k \pmod p,\ a \neq b\}, \\ S_{-k} &= \{u_a\ u_b \mid a + b \equiv -k \pmod p,\ a \neq b\}. \end{aligned}$$

Now for all $1 \leq k \leq r$, let

$$B_k = S_k \cup S_{-k} \cup \{u_{kr}\ u_{-kr}\}.$$

We can rewrite B_k as follows:

$$B_k = \begin{cases} (u_0\ u_k\ u_{-2k}\ u_{3k} \cdots u_{rk}\ u_{-rk}\ u_{(r+1)k} \cdots u_{2k}\ u_{-k}) & \text{if } r \text{ is odd}, \\ (u_0\ u_k\ u_{-2k}\ u_{3k} \cdots u_{-rk}\ u_{rk}\ u_{-(r+1)k} \cdots u_{2k}\ u_{-k}) & \text{if } r \text{ is even}. \end{cases}$$

It is easy to see that $\mathcal{G}_p$ and $\mathcal{B}_p$ are hamilton cycle decompositions of K_p; in fact, they are also symmetric [1]. Form the collections $\mathcal{G}'_p = \{G'_1, ..., G'_r\}$ and $\mathcal{B}'_p = \{B'_1, ..., B'_r\}$, where G'_k and B'_k are formed by removing the vertex u_0 and both of its incident edges from G_k and B_k, respectively, for $1 \leq k \leq r$. Clearly $\mathcal{G}'_p$ and $\mathcal{B}'_p$ are hamilton path decompositions of K_{p-1}; we show that they are orthogonal decompositions.

Theorem 3.1 *For every odd prime p, there exists a pair of MOSHLS$(p-1)$. Moreover, if $p \equiv 3 \pmod 4$ then there exists a pair of MOSHLS$(p-1)$ L and M such that L is diagonally distinguished with respect to M.*

Proof Let $\mathcal{G}'_p$ and $\mathcal{B}'_p$ be as defined in the preceding paragraph, and let $[r] = \{1, 2, ..., r\}$. For all $k \in [r]$, G'_k and B'_k share the endvertices u_k and u_{-k}. Moreover, $-r \equiv r + 1 \pmod p$, so G'_k and B'_k share the edge $u_{rk}\, u_{-rk}$, and $|E(G'_k) \cap E(B'_k)| \geq 1$. For all $k \neq \ell \in [r]$, there exists odd $m = 2q + 1 \in \mathbb{Z}_p$ such that $m\ell = \pm k$; since $k \neq \ell$, we know $m \neq 1$ and $q \neq 0$. G'_k and B'_ℓ share the edges $u_{q\ell}\, u_{-(q+1)\ell}$ and $u_{-q\ell}\, u_{(q+1)\ell}$, thus $|E(G'_k) \cap E(B'_\ell)| \geq 2$. Since $|E(G'_k) \cap E(B'_k)| \geq 1$ for all $k \in [r]$ and $|E(G'_k) \cap E(B'_\ell)| \geq 2$ for all $k \neq \ell \in [r]$, we must in fact have $|E(G'_k) \cap E(B'_k)| = 1$ for all $k \in [r]$ and $|E(G'_k) \cap E(B'_\ell)| = 2$ for all $k \neq \ell \in [r]$. Thus $\mathcal{G}'_p$ and $\mathcal{B}'_p$ are orthogonal hamilton path decompositions of K_{p-1}, and the corresponding double Latin squares form a pair of MOSHLS$(p-1)$.

Suppose $p \equiv 3 \pmod 4$, so that $(p-1)/2$ is odd, and let L and M be the squares corresponding to $\mathcal{G}'_p$ and $\mathcal{B}'_p$, respectively. For any $k \in [r]$, take a k-cell 2-coloring of L such that diagonal cells are colored red; this induces a 2-coloring of the path G'_k such that the first and last edges are colored blue. The nondiagonal cells containing k in both L and M are $(rk, -rk)$ and $(-rk, rk)$, so it suffices to show that the edge $u_{rk}\, u_{-rk}$ shared by G'_k and B'_k is also colored blue. The subpath $(u_0\, u_k\, u_{2k} \cdots u_{rk} u_{-rk})$ of G'_k is an odd path consisting of $(p-1)/2$ edges, so the last edge $u_{rk}\, u_{-rk}$ must receive the same color as the first edge, that is, blue. Thus L is diagonally distinguished with respect to M. □

4 | CONCLUSION

The following doubling construction was given in Hilton [9].

Theorem 4.1 (Theorem 5.9 in Hilton, [9]). *If there exists a pair of MOSHLS$(2n)$, then there exists a pair of MOSHLS$(4n)$.*

Let $X = \{1, 3, 5, 7, 9, 11, 13\}$; in Hilton [9] it was shown that a MOSHLS$(2n)$ exists for all $n = 2^\alpha m$, where $\alpha \geq 0$ and $m \in X$, by repeatedly applying Theorem 4.1 to a starting pair of MOSHLS$(2m)$. It is easy to check that for all $m \in X$, the starting pair of MOSHLS$(2m)$ obtained from what the authors call an *orthogonal m-procession* satisfies the diagonally distinguished 2-coloring property. Indeed, $2m + 1 \equiv 3 \pmod 4$ and corresponding paths from the respective decompositions share endvertices and a central edge, just as the decompositions given in Theorem 3.1.

Combining the results of Hilton [9] with Theorems 2.1 and 3.1, we can build a pair of MOSHLS$(2n)$ for many values of n by repeatedly applying Theorem 2.1 or 4.1 to a starting pair of MOSHLS obtained from Hilton [9] or Theorem 3.1. Let

$$Y_1 = \{(p-1)/2 \mid p \text{ is prime}, p \equiv 3 \pmod 4\} \cup \{7, 13\}$$

and

$$Y_2 = \{(p-1)/2 \mid p \text{ is prime}, p \equiv 1 \pmod 4\}.$$

We obtain the following result:

Theorem 4.2 *There exists a pair of MOSHLS*$(2n)$ *for all* $n = 2^{\alpha}3^{\beta}m_1$, *where* $\alpha, \beta \geq 0$ *and* $m_1 \in Y_1$, *and for all* $n = 2^{\gamma}m_2$, *where* $\gamma \geq 0$ *and* $m_2 \in Y_2$.

The set of values of $2n$ less than 60 for which the existence of a pair of MOSHLS$(2n)$ has not been determined is $\{34, 38, 50\}$.

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