Statement of Research Interests — Justin Z. Schroeder

My research is in *combinatorics*, with a primary focus on the interplay between graph theory and design theory and a secondary focus on graph labeling.

Graph theory is the study of binary relationships among collections of objects. For example, one might construct a graph with vertices representing airports and an edge between two vertices if there are direct flights available between those two airports; in this setting, the shortest path between two vertices represents the minimum number of layovers needed to fly from one airport to the other. Graphs arise naturally in the study of many structures (e.g. abelian groups, chemical molecules, and social media networks), and characterizing the properties of these graphs can offer insight on the nature and behavior of these structures.

Design theory is the study of systems of finite subsets that satisfy some given conditions. In general, design theorists attempt to answer two fundamental questions about a given set C of conditions. First, does there exist a system of finite subsets that satisfies C? Second, if the answer to the first question is yes, then how many such systems are there? These designs — which include such objects as latin squares, block designs, and difference sets — can be useful tools in other branches of pure and applied mathematics.

In the first two sections below, I describe some of my contributions to each of my main focus areas and present some ideas for future research. The third section addresses the suitability of some of these problems for undergraduate research projects.

1 Graph theory and design theory

One example of constructing a design from a graph G would be to take the collection of all cycles of length four in G. While for many graphs and constructions the resulting design is uninteresting, some special families of graphs give rise to designs with many nice properties (see Figure 1). Utilizing this connection, results for designs can be translated into results for graphs, and vice versa. Since designs are often easier to work with, this provides an effective tool for approaching many difficult problems in graph theory.

1.1 Graph embeddings

Roughly speaking, an *embedding* of a graph G is a drawing of G on some three-dimensional surface such that no two edges cross. Embeddings of certain graphs are the source of the so-called *Three Utilities Problem* (can you connect three houses to each of three different utilities without any utility lines crossing?) and the famous Four Color Theorem for maps.

A fundamental problem regarding embeddings is determining the *genus* of a given graph G: what is the least complicated surface S that admits an embedding of G? If you try to solve the Three Utilities Problem on a sphere, you will quickly find that it is impossible, but there does exist a solution on the torus, so the (orientable) genus of the underlying graph $K_{3,3}$ is one.

An ambitious approach to determining the genus of many graphs at once is to show that there exist graphs G and H such that G is a subgraph of H, but G and H have the same genus. If that is true, then any graph K satisfying $G \subset K \subset H$ must have the same genus as G and H. Such graphs G and H can easily be constructed if we have an embedding of a complete graph such that the boundary of every face is a hamilton cycle. Together with my advisor, Mark Ellingham, I was able to show in [1] that many such embeddings exist by using a recursive "tripling" construction that produced a hamilton cycle embedding of K_{3n-3} from a hamilton cycle embedding of K_n.

To make this tripling construction work, I needed to construct another family of embeddings: hamilton cycle embeddings of the complete tripartite graph $K_{n,n,n}$. The heart of my dissertation was building these hamilton cycle embeddings of $K_{n,n,n}$ for all n in both orientable and nonorientable surfaces. In the orientable case, this was accomplished by showing that these embeddings could be constructed from a pair of orthogonal latin squares if at least one of the squares satisfied an additional property on its entries. To find the required pair, I developed a new product construction that replaces the entries in a latin square of order m with permuted copies of a latin square of order n to form a latin square of order mn.

The problem of finding hamilton cycle embeddings of K_n for all permissible n, however, remains open. I recently opened a new line of inquiry that may lead us closer to a solution; namely, I have developed a set of conditions such that a latin square of order n satisfying these conditions corresponds to a hamilton cycle embedding of K_n. Using a computer, I have found such squares for $n \in \{11, 14, 15\}$, which covers the lowest previously unresolved cases.

Another family of designs closely related to latin squares are Steiner triple systems; a *Steiner triple system of order* n, briefly STS(n), is a pair (V, B) where V is a set of n elements and B is a collection of 3-subsets of V called blocks such that every pair of distinct elements is contained in precisely one block. The card game SET is based on an STS(81), and an STS(15) provides a solution to Kirkman's schoolgirl problem of 1850. For my undergraduate thesis I provided a new proof that an STS(n) exists if and only if $n \equiv$ 1 or 3 (mod 6).

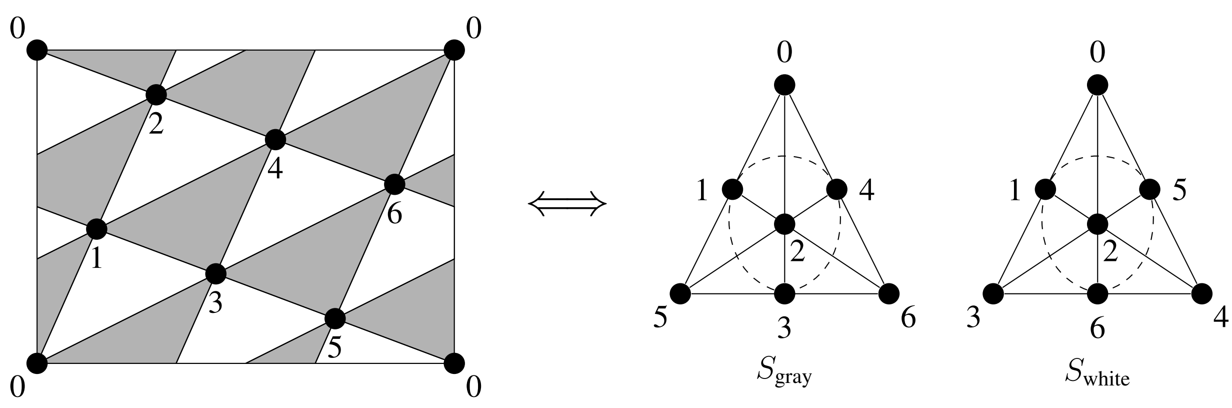

Figure 1: A triangulation of K_7 in the torus that corresponds to a biembedding of the two isomorphic STS(7)s S_{gray} and S_{white}.

Biembeddings of STS(n)s are triangulations of the complete graph K_n in which the faces can be colored gray and white such that the faces of each color form the blocks of an STS(n); see Figure 1 for a biembedding of two isomorphic STS(7)s. In [3] Grannell and I build biembeddings of STS(n)s that admit a special kind of automorphism, and in [5] McCourt and I build biembeddings in which both systems are isomorphic to the doubled affine STS(3^k). A problem left open in [5] that might be perfect for an undergraduate math or computer science major is finding a specific biembedding of STS(27)s in which one of the systems is the affine STS(27).

1.2 Graph decompositions

A *decomposition* $\mathcal{H}$ of a graph G is a set of edge-disjoint subgraphs $\mathcal{H} = \{H_1, ..., H_k\}$ that partition the edges of G. Frequently we desire these subgraphs to be either as large or as small as possible. For example, a Steiner triple system provides a decomposition of a complete graph into triangles, while a hamilton path decomposition of a graph G is a decomposition where each H_i is a path that uses every vertex in G.

One problem that has recently attracted my attention comes from [4]: does there exist a pair of orthogonal hamilton path decompositions of the complete graph K_{2n}? Hilton, et al., showed in [4] that the existence of such a decomposition is equivalent to the existence of a pair of mutually orthogonal symmetric hamiltonian double latin squares of order $2n$. I have already extended the results of the original paper by using a modification of the product construction for latin squares developed in my dissertation (see [7]), and lately I have seen even greater results by utilizing special structures from finite fields called Mullin-Nemeth starters.

Another special family of decompositions is called 1-factorizations. A *1-factor* is a collection of independent edges that together use every vertex of a graph, and a *1-factorization* of a graph G, then, is a decomposition $H_1, ..., H_k$ such that each H_i is a 1-factor. The goal of the Häggkvist problem is to find a 1-factorization of a graph G such that the union of any two 1-factors consists of only small cycles.

More specifically, let $l_F(n)$ represent the length of the longest cycle formed by any two 1-factors in a 1-factorization F of the complete bipartite graph $K_{n,n}$, and take $l(n)$ to be the minimum of $l_F(n)$ over all 1-factors F of $K_{n,n}$. The Häggkvist problem asks, how small is $l(n)$? It is conjectured that $l(n) \leq 6$ for all n, but it is only known for a few specific families of n. I have unpublished work using Steiner triple systems and latin squares to show $l(n) \leq 12$ for many values of n, and I believe this latter bound can be established for all n using tools from design theory.

2 Graph labeling

Given a graph G with n vertices, can we assign the values $1, 2, ..., n$ to the vertices of G such that the labels on adjacent vertices are always relatively prime? If so, such a labeling is called a *prime labeling*, and the conjecture that initiated research into this type of labeling—that a union of cycles has a prime labeling if and only if at most one cycle has odd length—is still open and a fruitful area of research for students that want to combine graph theory with number theory. I have previously worked with a group of undergraduate students on a project in this area, and we showed in [6] that the generalized Petersen graph $G(n, k)$ has a prime labeling for many pairs (n, k) where n is even and k is odd (if n is odd or k is even then no such labeling exists).

An idea that is closely tied to graph labeling is symmetry breaking, which is usually done by imposing further structure on a graph so that no automorphisms of G preserve this additional structure. For example, we can try to partition the vertices of G into subsets (equivalent to assigning set-labels to the vertices) so that no automorphism of G preserves this set structure. Ellingham and I introduced this idea in [2], and proved (with three small exceptions) that the complete multipartite graph $K_{m(n)} = K_{n,n,...,n}$ has a distinguishing partition if and only if $m \geq \lfloor \log_2(n + 1)\rfloor + 2$. This result was obtained by constructing asymmetric n-uniform hypergraphs with m edges and converting these into partitions of the complete multipartite graph $K_{m(n)}$.

3 Suitability of my research interests for undergraduates

Problems in combinatorics and graph theory have a unique advantage that help make them perfect candidates for undergraduate research projects: even though their solutions may be quite difficult, the problems are usually easy to explain and visualize. In fact, I have started explaining some graph theory ideas to my children (ages 5 and 7), and they have been able to complete some selected problems from the early sections of an undergraduate-level graph theory textbook. Because the problems are intuitive and easy to comprehend, students are able to focus their energy on *solving* the problem, instead of merely trying to understand it.

In addition to the problems already mentioned in Sections 1 and 2, I have several other questions that would be approachable by a student interested in math and/or computer science. Can my rudimentary algorithm for finding latin squares that correspond to hamilton cycle embeddings of complete graphs K_n be improved, allowing us to find solutions for open cases where $n \geq 19$? Is there an STS(25) that gives us a 1-factorization of $K_{25,25}$ where the union of any two 1-factors contains cycles of maximum length 12? What number theoretic conjectures would need to be proved to guarantee that the union of even cycles always has a prime labeling? I am excited to start leading groups of undergraduate students that want to explore these questions—and raise some new ones!

References

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